

# Essential Mathematics

## for Economic Analysis

Knut Sydsæter, Peter Hammond,  
Arne Strøm & Andrés Carvajal



Pearson

Sixth Edition



ESSENTIAL MATHEMATICS FOR

**ECONOMIC  
ANALYSIS**



Pearson

At Pearson, we have a simple mission: to help people make more of their lives through learning.

We combine innovative learning technology with trusted content and educational expertise to provide engaging and effective learning experiences that serve people wherever and whenever they are learning.

From classroom to boardroom, our curriculum materials, digital learning tools and testing programmes help to educate millions of people worldwide – more than any other private enterprise.

Every day our work helps learning flourish, and wherever learning flourishes, so do people.

To learn more, please visit us at [www.pearson.com/uk](http://www.pearson.com/uk)

ESSENTIAL MATHEMATICS FOR

# ECONOMIC ANALYSIS

SIXTH EDITION

Knut Sydsæter, Peter Hammond,  
Arne Strøm and Andrés Carvajal



---

Harlow, England • London • New York • Boston • San Francisco • Toronto • Sydney • Dubai • Singapore • Hong Kong  
Tokyo • Seoul • Taipei • New Delhi • Cape Town • São Paulo • Mexico City • Madrid • Amsterdam • Munich • Paris • Milan

**PEARSON EDUCATION LIMITED**

KAO Two  
KAO Park  
Harlow CM17 9NA  
United Kingdom  
Tel: +44 (0)1279 623623  
Web: [www.pearson.com/uk](http://www.pearson.com/uk)

---

First published by Prentice Hall, Inc. 1995 (print)  
Second edition published 2006 (print)  
Third edition published 2008 (print)  
Fourth edition published by Pearson Education Limited 2012 (print)  
Fifth edition published 2016 (print and electronic)

© Prentice Hall, Inc. 1995 (print)  
© Knut Sydsæter, Peter Hammond, Arne Strøm and Andrés Carvajal 2016 (print and electronic)  
© Knut Sydsæter, Peter Hammond, Arne Strøm and Andrés Carvajal 2021 (print and electronic)

The rights of Knut Sydsæter, Peter Hammond, Arne Strøm and Andrés Carvajal to be identified as authors of this work has been asserted by them in accordance with the Copyright, Designs and Patents Act 1988.

The print publication is protected by copyright. Prior to any prohibited reproduction, storage in a retrieval system, distribution or transmission in any form or by any means, electronic, mechanical, recording or otherwise, permission should be obtained from the publisher or, where applicable, a licence permitting restricted copying in the United Kingdom should be obtained from the Copyright Licensing Agency Ltd, Barnard's Inn, 86 Fetter Lane, London EC4A 1EN.

The ePublication is protected by copyright and must not be copied, reproduced, transferred, distributed, leased, licensed or publicly performed or used in any way except as specifically permitted in writing by the publishers, as allowed under the terms and conditions under which it was purchased, or as strictly permitted by applicable copyright law. Any unauthorised distribution or use of this text may be a direct infringement of the authors' and the publisher's rights and those responsible may be liable in law accordingly.

Pearson Education is not responsible for the content of third-party internet sites.

ISBN: 978-1-292-35928-1 (print)  
978-1-292-35929-8 (PDF)  
978-1-292-35932-8 (ePub)

**British Library Cataloguing-in-Publication Data**

A catalogue record for the print edition is available from the British Library

**Library of Congress Cataloging-in-Publication Data**

Names: Sydsæter, Knut, author. | Hammond, Peter J., 1945- author.  
Title: Essential mathematics for economic analysis / Knut Sydsæter, Peter Hammond, Arne Strøm and Andrés Carvajal.  
Description: Sixth edition. | Hoboken , NJ: Pearson, 2021. | Includes bibliographical references and index. | Summary: "The subject matter that modern economics students are expected to master makes significant mathematical demands. This is true even of the less technical "applied" literature that students will be expected to read for courses in fields such as public finance, industrial organization, and labour economics, amongst several others. Indeed, the most relevant literature typically presumes familiarity with several important mathematical tools, especially calculus for functions of one and several variables, as well as a basic understanding of multivariable optimization problems with or without constraints. Linear algebra is also used to some extent in economic theory, and a great deal more in econometrics" – Provided by publisher.  
Identifiers: LCCN 2021006079 (print) | LCCN 2021006080 (ebook) | ISBN 9781292359281 (paperback) | ISBN 9781292359298 (pdf) | ISBN 9781292359328 (epub)  
Subjects: LCSH: Economics, Mathematical.  
Classification: LCC HB135 .S886 2021 (print) | LCC HB135 (ebook) | DDC 330.01/51–dc23  
LC record available at <https://lcn.loc.gov/2021006079>  
LC ebook record available at <https://lcn.loc.gov/2021006080>

10 9 8 7 6 5 4 3 2 1  
25 24 23 22 21

Cover design by Michelle Morgan, At the Pop Ltd.

Front cover image © yewkeo/iStock/Getty Images Plus

Print edition typeset in 10/13pt TimesLTPro by SPi Global  
Printed and bound by L.E.G.O. S.p.A., Italy

NOTE THAT ANY PAGE CROSS REFERENCES REFER TO THE PRINT EDITION

*To Knut Sydsæter (1937–2012), an inspiring mathematics teacher, as well as wonderful friend and colleague, whose vision, hard work, high professional standards, and sense of humour were all essential in creating this book.*

—Arne, Peter and Andrés

*To Else, my loving and patient wife.*

—Arne

*To the memory of my parents Elsie (1916–2007) and Fred (1916–2008), my first teachers of Mathematics, basic Economics, and many more important things.*

—Peter

*To Yeye and Tata, my best ever students of “matemáquinas”, who wanted this book to start with “Once upon a time ...”. E para a Pipoca, com amor infinito à infinito.*

—Andrés



# CONTENTS

Preface	xiii		
<b>I PRELIMINARIES</b>	<b>1</b>		
<b>1 Essentials of Logic and Set Theory</b>	<b>3</b>	<b>3 Solving Equations</b>	<b>77</b>
1.1 Essentials of Set Theory	3	3.1 Solving Equations	77
1.2 Essentials of Logic	10	3.2 Equations and Their Parameters	80
1.3 Mathematical Proofs	16	3.3 Quadratic Equations	83
1.4 Mathematical Induction	18	3.4 Some Nonlinear Equations	89
Review Exercises	20	3.5 Using Implication Arrows	91
		3.6 Two Linear Equations in Two Unknowns	92
		Review Exercises	96
<b>2 Algebra</b>	<b>23</b>	<b>4 Functions of One Variable</b>	<b>99</b>
2.1 The Real Numbers	23	4.1 Introduction	99
2.2 Integer Powers	26	4.2 Definitions	100
2.3 Rules of Algebra	33	4.3 Graphs of Functions	106
2.4 Fractions	38	4.4 Linear Functions	110
2.5 Fractional Powers	43	4.5 Linear Models	116
2.6 Inequalities	48	4.6 Quadratic Functions	120
2.7 Intervals and Absolute Values	52	4.7 Polynomials	127
2.8 Sign Diagrams	56	4.8 Power Functions	135
2.9 Summation Notation	59	4.9 Exponential Functions	136
2.10 Rules for Sums	62	4.10 Logarithmic Functions	141
2.11 Newton's Binomial Formula	66	Review Exercises	147
2.12 Double Sums	70	<b>5 Properties of Functions</b>	<b>151</b>
Review Exercises	72	5.1 Shifting Graphs	151
		5.2 New Functions from Old	156



5.3	Inverse Functions	160	8.5	Second-Derivative Tests	312
5.4	Graphs of Equations	166	8.6	Inflection Points	316
5.5	Distance in the Plane	170		Review Exercises	319
5.6	General Functions	174			
	Review Exercises	177			
<b>II SINGLE VARIABLE CALCULUS</b>		<b>179</b>	<b>9 Optimization</b>		<b>321</b>
<b>6 Differentiation</b>		<b>181</b>	9.1	Extreme Points	321
6.1	Slopes of Curves	181	9.2	Simple Tests for Extreme Points	324
6.2	Tangents and Derivatives	183	9.3	Economic Examples	328
6.3	Increasing and Decreasing Functions	189	9.4	The Extreme and Mean Value Theorems	334
6.4	Economic Applications	192	9.5	Further Economic Examples	340
6.5	A Brief Introduction to Limits	195	9.6	Local Extreme Points	345
6.6	Simple Rules for Differentiation	201		Review Exercises	352
6.7	Sums, Products, and Quotients	205	<b>10 Integration</b>		<b>355</b>
6.8	The Chain Rule	212	10.1	Indefinite Integrals	355
6.9	Higher-Order Derivatives	218	10.2	Area and Definite Integrals	361
6.10	Exponential Functions	220	10.3	Properties of Definite Integrals	368
6.11	Logarithmic Functions	224	10.4	Economic Applications	373
	Review Exercises	230	10.5	Integration by Parts	380
<b>7 Derivatives in Use</b>		<b>233</b>	10.6	Integration by Substitution	384
7.1	Implicit Differentiation	233	10.7	Improper Integrals	389
7.2	Economic Examples	240		Review Exercises	396
7.3	The Inverse Function Theorem	244	<b>11 Topics in Finance and Dynamics</b>		<b>399</b>
7.4	Linear Approximations	248	11.1	Interest Periods and Effective Rates	399
7.5	Polynomial Approximations	253	11.2	Continuous Compounding	403
7.6	Taylor's Formula	256	11.3	Present Value	405
7.7	Elasticities	259	11.4	Geometric Series	408
7.8	Continuity	264	11.5	Total Present Value	414
7.9	More on Limits	270	11.6	Mortgage Repayments	419
7.10	The Intermediate Value Theorem	279	11.7	Internal Rate of Return	423
7.11	Infinite Sequences	283	11.8	A Glimpse at Difference Equations	425
7.12	L'Hôpital's Rule	287	11.9	Essentials of Differential Equations	428
	Review Exercises	292	11.10	Separable and Linear Differential Equations	435
<b>8 Concave and Convex Functions</b>		<b>295</b>		Review Exercises	441
8.1	Intuition	295	<b>III MULTIVARIABLE ALGEBRA</b>		<b>445</b>
8.2	Definitions	297	<b>12 Matrix Algebra</b>		<b>447</b>
8.3	General Properties	305	12.1	Matrices and Vectors	447
8.4	First-Derivative Tests	309			

12.2	Systems of Linear Equations	450	<b>15 Partial Derivatives in Use</b>	<b>613</b>	
12.3	Matrix Addition	453	15.1	A Simple Chain Rule	613
12.4	Algebra of Vectors	455	15.2	Chain Rules for Many Variables	619
12.5	Matrix Multiplication	458	15.3	Implicit Differentiation along a Level Curve	623
12.6	Rules for Matrix Multiplication	463	15.4	Level Surfaces	628
12.7	The Transpose	470	15.5	Elasticity of Substitution	632
12.8	Gaussian Elimination	473	15.6	Homogeneous Functions of Two Variables	634
12.9	Geometric Interpretation of Vectors	479	15.7	Homogeneous and Homothetic Functions	639
12.10	Lines and Planes	487	15.8	Linear Approximations	645
	Review Exercises	492	15.9	Differentials	654
<b>13</b>	<b>Determinants, Inverses, and Quadratic Forms</b>	<b>495</b>	15.10	Systems of Equations	659
13.1	Determinants of Order 2	495	15.11	Differentiating Systems of Equations	663
13.2	Determinants of Order 3	499		Review Exercises	669
13.3	Determinants in General	504	<b>16 Multiple Integrals</b>	<b>673</b>	
13.4	Basic Rules for Determinants	508	16.1	Double Integrals Over Finite Rectangles	673
13.5	Expansion by Cofactors	514	16.2	Infinite Rectangles of Integration	680
13.6	The Inverse of a Matrix	517	16.3	Discontinuous Integrands and Other Extensions	682
13.7	A General Formula for the Inverse	524	16.4	Integration Over Many Variables	684
13.8	Cramer's Rule	527	<b>V MULTIVARIABLE OPTIMIZATION</b>	<b>687</b>	
13.9	The Leontief Model	531	<b>17 Unconstrained Optimization</b>	<b>689</b>	
13.10	Eigenvalues and Eigenvectors	536	17.1	Two Choice Variables: Necessary Conditions	690
13.11	Diagonalization	543	17.2	Two Choice Variables: Sufficient Conditions	694
13.12	Quadratic Forms	547	17.3	Local Extreme Points	699
	Review Exercises	556	17.4	Linear Models with Quadratic Objectives	705
<b>IV MULTIVARIABLE CALCULUS</b>	<b>559</b>		17.5	The Extreme Value Theorem	712
<b>14 Functions of Many Variables</b>	<b>561</b>		17.6	Functions of More Variables	717
14.1	Functions of Two Variables	561	17.7	Comparative Statics and the Envelope Theorem	722
14.2	Partial Derivatives with Two Variables	565		Review Exercises	728
14.3	Geometric Representation	571			
14.4	Surfaces and Distance	578			
14.5	Functions of $n$ Variables	581			
14.6	Partial Derivatives with Many Variables	586			
14.7	Convex Sets	590			
14.8	Concave and Convex Functions	595			
14.9	Economic Applications	606			
14.10	Partial Elasticities	608			
	Review Exercises	610			

<b>18 Equality Constraints</b>	<b>731</b>	<b>20 Nonlinear Programming</b>	<b>797</b>
18.1 The Lagrange Multiplier Method	731	20.1 Two Variables and One Constraint	797
18.2 Interpreting the Lagrange Multiplier	739	20.2 Many Variables and Inequality Constraints	804
18.3 Multiple Solution Candidates	742	20.3 Nonnegativity Constraints	812
18.4 Why Does the Lagrange Multiplier Method Work?	744	Review Exercises	817
18.5 Sufficient Conditions	749	<b>Appendix</b>	<b>819</b>
18.6 Additional Variables and Constraints	753	Geometry	819
18.7 Comparative Statics	759	The Greek Alphabet	822
Review Exercises	765	Bibliography	822
<b>19 Linear Programming</b>	<b>769</b>	<b>Solutions to the Exercises</b>	<b>823</b>
19.1 A Graphical Approach	770	<b>Index</b>	<b>943</b>
19.2 Introduction to Duality Theory	776	<b>Publisher's Acknowledgements</b>	<b>953</b>
19.3 The Duality Theorem	781		
19.4 A General Economic Interpretation	786		
19.5 Complementary Slackness	788		
Review Exercises	794		

**Supporting resources**

Visit [go.pearson.com/uk/he/resources](http://go.pearson.com/uk/he/resources) to find valuable online resources

**For students**

- A new Student's Manual provides more detailed solutions to the problems marked (SM) in the book

**For instructors**

- The fully updated Instructor's Manual provides instructors with a collection of problems that can be used for tutorials and exams

**In MyLab Maths you will find:**

- A study plan, which creates a personalised learning path through resources, based on diagnostic tests;
- A wealth of questions and activities, including algorithmic, graphing and multiple-choice questions and exercises;
- Pearson eText, an online version of your textbook which you can take with you anywhere and which allows you to add notes, highlights, bookmarks and easily search the text;
- If assigned by your lecturer, additional homework, quizzes and tests.

**Lecturers additionally will find:**

- A powerful gradebook, which tracks student performance and can highlight areas where the whole class needs more support;
- Additional questions and exercises, which can be assigned to students as homework, quizzes and tests, freeing up the time in class to take learning further;
- Lecturer resources, which support the delivery of the module;
- Communication tools, such as discussion boards, to drive student engagement.



# PREFACE

*Once upon a time there was a sensible straight line who was hopelessly in love with a dot. 'You're the beginning and the end, the hub, the core and the quintessence,' he told her tenderly, but the frivolous dot wasn't a bit interested, for she only had eyes for a wild and unkempt squiggle who never seemed to have anything on his mind at all. All of the line's romantic dreams were in vain, until he discovered . . . angles! Now, with newfound self-expression, he can be anything he wants to be—a square, a triangle, a parallelogram . . . And that's just the beginning!*

—Norton Juster (*The Dot and the Line: A Romance in Lower Mathematics* 1963)

*I came to the position that mathematical analysis is not one of many ways of doing economic theory: It is the only way. Economic theory is mathematical analysis. Everything else is just pictures and talk.*

—R. E. Lucas, Jr. (2001)

## Purpose

The subject matter that modern economics students are expected to master makes significant mathematical demands. This is true even of the less technical “applied” literature that students will be expected to read for courses in fields such as public finance, industrial organization, and labour economics, amongst several others. Indeed, the most relevant literature typically presumes familiarity with several important mathematical tools, especially calculus for functions of one and several variables, as well as a basic understanding of multivariable optimization problems with or without constraints. Linear algebra is also used to some extent in economic theory, and a great deal more in econometrics.

The purpose of *Essential Mathematics for Economic Analysis*, therefore, is to help economics students acquire enough mathematical skill to access the literature that is most relevant to their undergraduate study. This should include what some students will need to conduct successfully an undergraduate research project or honours thesis.


As the title suggests, this is a book on *mathematics*, whose material is arranged to allow progressive learning of mathematical topics. That said, we do frequently emphasize economic applications, many of which are listed on the inside front cover. These not only

help motivate particular mathematical topics; we also want to help prospective economists acquire mutually reinforcing intuition in both mathematics and economics. Indeed, as the list of examples on the inside front cover suggests, a considerable number of economic concepts and ideas receive some attention.

We emphasize, however, that this is not a book about economics or even about mathematical economics. Students should learn economic theory systematically from other courses, which use other textbooks. We will have succeeded if they can concentrate on the economics in these courses, having already thoroughly mastered the relevant mathematical tools this book presents.

## Special Features and Accompanying Material

Virtually all sections of the book conclude with exercises, often quite numerous. There are also many review exercises at the end of each chapter. Solutions to almost all these exercises are provided at the end of the book, sometimes with several steps of the answer laid out.

There are two main sources of supplementary material. The first, for both students and their instructors, is via MyLab. Students who have arranged access to this web site for our book will be able to generate a practically unlimited number of additional problems which test how well some of the key ideas presented in the text have been understood. More explanation of this system is offered after this preface. The same web page also has a “student resources” tab with access to a *Student’s Manual* with more extensive answers (or, in the case of a few of the most theoretical or difficult problems in the book, the only answers) to problems marked with the special symbol .

The second source, for instructors who adopt the book for their course, is an *Instructor’s Manual* that may be downloaded from the publisher’s Instructor Resource Centre.

In addition, for courses with special needs, there is a brief online appendix on trigonometric functions and complex numbers. This is also available via MyLab.

## Prerequisites

Experience suggests that it is quite difficult to start a book like this at a level that is really too elementary.<sup>1</sup> These days, in many parts of the world, students who enter college or university and specialize in economics have an enormous range of mathematical backgrounds and aptitudes. These range from, at the low end, a rather shaky command of elementary algebra, up to real facility in the calculus of functions of one variable. Furthermore, for many economics students, it may be some years since their last formal mathematics course. Accordingly, as mathematics becomes increasingly essential for specialist studies in economics, we feel obliged to provide as much quite elementary material as is reasonably possible. Our aim here is to give those with weaker mathematical backgrounds the chance to get started, and even to acquire a little confidence with some easy problems they can really solve on their own.

---

<sup>1</sup> In a recent test for 120 first-year students intending to take an elementary economics course, there were 35 different answers to the problem of expanding  $(a + 2b)^2$ .

To help instructors judge how much of the elementary material students really know before starting a course, the *Instructor's Manual* provides some diagnostic test material. Although each instructor will obviously want to adjust the starting point and pace of a course to match the students' abilities, it is perhaps even more important that each individual student appreciates his or her own strengths and weaknesses, and receives some help and guidance in overcoming any of the latter. This makes it quite likely that weaker students will benefit significantly from the opportunity to work through the early more elementary chapters, even if they may not be part of the course itself.

As for our economic discussions, students should find it easier to understand them if they already have a certain very rudimentary background in economics. Nevertheless, the text has often been used to teach mathematics for economics to students who are studying elementary economics at the same time. Nor do we see any reason why this material cannot be mastered by students interested in economics before they have begun studying the subject in a formal university course.

## Topics Covered

After the introductory material in Chapters 1 to 3, a fairly leisurely treatment of standard single variable differential calculus is contained in Chapters 4 to 7. This is followed by Chapter 8 on concave and convex functions, by Chapter 9 on optimization, Chapter 10 on integration, and then by some basic financial models as well as difference and differential equations in Chapter 11. This may be as far as some elementary courses will go. Students who already have a thorough grounding in single variable calculus, however, may only need to go fairly quickly over some special topics in these chapters such as elasticity and conditions for global optimization that are often not thoroughly covered in standard calculus courses.

We have already suggested the importance for budding economists of the algebra of matrices and determinants (Chapters 12 and 13), of multivariable calculus (Chapters 14–16), and of optimization theory with and without constraints (Chapters 17–20). These last nine chapters in some sense represent the heart of the book, on which students with a thorough grounding in single variable calculus can probably afford to concentrate.

## Satisfying Diverse Requirements

The less ambitious student can concentrate on learning the key concepts and techniques of each chapter. Often, these appear boxed and/or in colour, in order to emphasize their importance. Problems are essential to the learning process, and the easier ones should definitely be attempted. These basics should provide enough mathematical background for the student to be able to understand much of the economic theory that is embodied in applied work at the advanced undergraduate level.

Students who are more ambitious, or who are led on by more demanding teachers, can try the more difficult problems. They can also study the more technical material which is intended to encourage students to ask why a result is true, or why a problem should be tackled in a particular way. If more readers gain at least a little additional mathematical insight from working through these more challenging parts of our book, so much the better.



The most able students, especially those intending to undertake postgraduate study in economics or some related subject, will benefit from a fuller explanation of some topics than we have been able to provide here. On a few occasions, therefore, we take the liberty of referring to our more advanced companion volume, *Further Mathematics for Economic Analysis* (usually abbreviated to FMEA). This is written jointly with our colleague Atle Seierstad in Oslo. In particular, FMEA offers a proper treatment of topics like systems of difference and differential equations, as well as dynamic optimization, that we think go rather beyond what is really “essential” for all economics students.

## Changes in the Fourth Edition

We have been gratified by the number of students and their instructors from many parts of the world who appear to have found the first three editions useful.<sup>2</sup> We have accordingly been encouraged to revise the text thoroughly once again. There are numerous minor changes and improvements, including the following in particular:

1. The main new feature is MyMathLab Global,<sup>3</sup> explained on the page after this preface, as well as on the back cover.
2. New exercises have been added for each chapter.
3. Some of the figures have been improved.

## Changes in the Fifth Edition

The most significant change in this edition is that, tragically, we have lost the main author and instigator of this project. Our good friend and colleague Knut Sydsæter died suddenly on 29th September 2012, while on holiday in Spain with his wife Malinka Staneva, a few days before his 75th birthday. An obituary written by Jens Stoltenberg, at that time the Prime Minister of Norway, includes this tribute to Knut’s skills as one of his teachers:

With a small sheet of paper as his manuscript he introduced me and generations of other economics students to mathematics as a tool in the subject of economics. With professional weight, commitment, and humour, he was both a demanding and an inspiring lecturer. He opened the door into the world of mathematics. He showed that mathematics is a language that makes it possible to explain complicated relationships in a simple manner.

At a web page that hosts a copy of this obituary one can also find other tributes to Knut, including some recollections of how previous editions of this book came to be written.<sup>4</sup>

Despite losing Knut as its main author, it was clear that this book needed to be kept alive, following desires that Knut himself had often expressed while he was still with us.

---

<sup>2</sup> Different English versions of this book have been translated into Albanian, French, German, Hungarian, Italian, Portuguese, Spanish, and Turkish.

<sup>3</sup> Superseded by MyLab for this sixth edition.

<sup>4</sup> See <https://web.stanford.edu/~hammond/sydsaeter.html>

Fortunately, it had already been agreed that the team of co-authors should be joined by Andrés Carvajal, a former colleague of Peter's at Warwick who, at the time of preparing the Fifth Edition, had just joined the University of California at Davis. Andrés had already produced a new Spanish version of the previous edition of this book; he has now become a co-author of this latest English version. It is largely on his initiative that we have taken the important step of extensively rearranging the material in the first three chapters in a more logical order, with set theory now coming first.

The other main change is one that we hope is invisible to the reader. Previous editions had been produced using the “plain  $\text{T}_{\text{E}}\text{X}$ ” typesetting system that dates back to the 1980s, along with some ingenious macros that Arne had devised in collaboration with Arve Michaelsen of the Norwegian typesetting firm Matematisk Sats. For technical reasons we decided that the new edition had to be produced using the enrichment of plain  $\text{T}_{\text{E}}\text{X}$  called  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  that has by now become the accepted international standard for typesetting mathematical material. We have therefore attempted to adapt and extend some standard  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  packages in order to preserve as many good features as possible of our previous editions.

## Changes in the Sixth Edition

For this sixth edition, the surviving authors decided to rearrange the chapters considerably. Recent previous editions included a chapter on linear programming, which was deferred until after the two chapters on matrix algebra. Yet the key idea of complementary slackness had arisen previously in an earlier chapter on nonlinear programming. So we have moved matrix algebra much further forward, so that it precedes multivariate calculus. This allows new tools to be used in our treatment of multivariate calculus, and subsequently in the last four chapters that are now devoted exclusively to optimization.

Not only have the existing chapters been rearranged, however. We have increased their number from 17 to 20. This is partly because the chapter on constrained optimization has been split into two. The first part dealing with equality constraints now comes in Chapter 18, before Chapter 19 on linear programming, including its discussion of complementary slackness. The last part of the earlier chapter on inequality constraints is now the separate Chapter 20.

The other two extra chapters are new. Chapter 8 considers concave and convex functions of one variable, including results on supergradients of concave functions and subgradients of convex functions that play a key role in the theory of optimization. Later chapters extend some of these results to functions of 2 and then  $n$  variables. There is also a brief chapter (16) on multiple integrals.

Finally, we mention significant additions to Chapter 13 that consider eigenvalues and quadratic forms. These additions allow a more extensive treatment, based on the Hessian matrix, of second-order conditions for, in Chapter 15, a function of several variables to be concave, and in Chapter 17, for a critical point to be a maximum or minimum. As a result, we can provide a somewhat better discussion in Chapter 20 of how, for the case of concave programming problems, the Karush–Kuhn–Tucker conditions provide sufficient conditions for an optimal point.

## Other Acknowledgements

Over the years we have received help from so many colleagues, lecturers at other institutions, and students, that it is impractical to mention them all.

Andrés Carvajal is indebted to: Yiqian Zhao and Xinhui Yang, for all their great work in the revision of the material for this edition; Professor Janine Wilson for encouraging him in the idea that the more economic applications the book contains, the better is the mathematical explanation; Professor Jim Wiseman, for his feedback on the previous edition and for sharing his views on how it could be improved; and to the following UC Davis students who patiently went over different chapters, fishing for mistakes and making sure that all was well: Xinghe Bai, Veronica Contreras, Nathan Gee, Anjali Khalasi, Yannan Li, Daniel Scates, Kelly Stangl, and Yiping Su.

As in previous editions of this book, we are very happy to acknowledge with gratitude the encouragement and assistance of our contacts at Pearson. For this sixth edition, these include Catherine Yates (Product Manager) and Melanie Carter (Senior Content Producer). We were also glad to be able to work successfully with Vivek Khandelwal of SPi Global, who was in charge of the typesetting, and Lou Attwood of SpacedEns Editorial Services, who assisted us with proof-reading. All were very helpful and attentive in answering our frequent e-mails in a friendly and encouraging way, while making sure that this new edition really is getting into print in a timely manner.

On the more academic side, very special thanks go to Prof. Dr Fred Böker at the University of Göttingen. He is not only responsible for translating several previous editions of this book into German, but has also shown exceptional diligence in paying close attention to the mathematical details of what he was translating. We appreciate the resulting large number of valuable suggestions for improvements and corrections that he has continued to provide, sometimes at the instigation of Dr Egle Tafenau, who was also using the German version of our textbook in her teaching.

We are also grateful to Kenneth Judd of the Hoover Institution at Stanford for taking the trouble to persuade us that we should follow what has become the standard practice of attaching the name of William Karush, along with those of Harold Kuhn and Albert Tucker, to the key “KKT conditions” presented in Chapter 20 for solving a nonlinear programming problem with inequality constraints.

Thanks too, to Dr Mauro Bambi at Durham University for creating and curating question content for MyLab Maths, and to Professor Carsten Berthram Haahr Andersen at Aarhus University, Denmark for his feedback on the MyLab.

To these and all the many unnamed persons and institutions who have helped us make this text possible, including some whose anonymous comments on earlier editions were forwarded to us by the publisher, we would like to express our deep appreciation and gratitude. We hope that all those who have assisted us may find the resulting product of benefit to their students. This, we can surely agree, is all that really matters in the end.

*Andrés Carvajal, Peter Hammond, and Arne Strøm*

Davis, Coventry, and Oslo, January 2021

I

# PRELIMINARIES



# 1

# ESSENTIALS OF LOGIC AND SET THEORY

*It is clear that economics, if it is to be a science at all, must be a mathematical science.*  
—William Stanley Jevons<sup>1</sup>

Arguments in mathematics require tight logical reasoning, and arguments in modern economic analysis are no exception to this rule. It is useful for us, then, to present some basic concepts from logic, as well as a brief section on mathematical proofs.

We precede this with a short introduction to set theory. This is useful not just for its importance in mathematics, but also because of a key role that sets play in economics: in most economic models, it is assumed that economic agents pursue some specific goal like profit, and make an optimal choice from a specified feasible set of alternatives.

The chapter winds up with a discussion of mathematical induction. Occasionally, this method is used directly in economic arguments; more often, it is needed to understand mathematical results which economists use.

## 1.1 Essentials of Set Theory

In daily life, we constantly group together objects of the same kind. For instance, the faculty of a university signifies all the members of its academic staff. A garden refers to all the plants that are growing in it. An economist may talk about all Scottish firms with over 300 employees, or all taxpayers in Germany who earned between €50 000 and €100 000 in 2019. Or suppose a student who is planning what combination of laptop and smartphone to buy for use in college. The student may consider all combinations whose total price does not exceed what she can afford. In all these cases, we have a collection of objects that we may want to view as a whole. In mathematics, such a collection is called a *set*, and the objects that belong to the set are called its *elements*, or its *members*.

---

<sup>1</sup> *The Theory of Political Economy* (1871)

The simplest way to specify a set is to list its members, in any order, between the opening brace { and the closing brace }. An example is the set whose members are the first three letters in the English alphabet,  $S = \{a, b, c\}$ . Or it might be a set consisting of three members represented by the letters  $a, b$ , and  $c$ . For example, if  $a = 0$ ,  $b = 1$ , and  $c = 2$ , then  $S = \{0, 1, 2\}$ . Also,  $S = \{a, b, c\}$  denotes the set of roots of the cubic equation  $(x - a)(x - b)(x - c) = 0$  in the unknown  $x$ , where  $a, b$ , and  $c$  are any three real numbers. Verbally, the braces are read as “the set consisting of”.

Since a set is fully specified by listing all its members, two sets  $A$  and  $B$  are considered *equal* if they contain exactly the same elements: each element of  $A$  is an element of  $B$ ; conversely, each element of  $B$  is an element of  $A$ . In this case, we write  $A = B$ . Consequently,  $\{1, 2, 3\} = \{3, 2, 1\}$ , because the order in which the elements are listed has no significance; and  $\{1, 1, 2, 3\} = \{1, 2, 3\}$ , because a set is not changed if some elements are listed more than once.

The symbol “ $\emptyset$ ” denotes the set that has no elements. It is called the *empty set*. Note that it is *the*, and not *an*, empty set. This is so, following the principle that a set is completely defined by listing all its members: there can only be one set that contains no elements. The empty set is the same, whether it is being studied by a child in elementary school who thinks about cows that can jump over the moon, or by a physicist at CERN who thinks about subatomic particles that move faster than the speed of light—or, indeed, by an economics student reading this book!

## Specifying a Property

Not every set can be defined by listing all its members, however. For one thing, some sets are infinite—that is, they contain infinitely many members. Such infinite sets are rather common in economics. Take, for instance, the budget set that arises in consumer theory. Suppose there are two goods with quantities denoted by  $x$  and  $y$ . Suppose these two goods can be bought at prices per unit that equal  $p$  and  $q$ , respectively. A consumption bundle is a pair of quantities of the two goods,  $(x, y)$ . Its value at prices  $p$  and  $q$  is  $px + qy$ . Suppose that a consumer has an amount  $m$  to spend on the two goods. Then the *budget constraint* is  $px + qy \leq m$ , assuming that the consumer is free to underspend. If one also accepts that the quantity consumed of each good must be nonnegative, then the *budget set*, which will be denoted by  $B$ , consists of all those consumption bundles  $(x, y)$  satisfying the three inequalities  $px + qy \leq m$ ,  $x \geq 0$ , and  $y \geq 0$ . This set is illustrated in Fig. 4.4.12. Standard notation for it is

$$B = \{(x, y) : px + qy \leq m, x \geq 0, y \geq 0\} \quad (1.1.1)$$

The two braces { and } are still used to denote “the set consisting of”. However, instead of listing all the members, which is impossible for the infinite set of points in the triangular budget set  $B$ , it is specified in two parts. First, before the colon,  $(x, y)$  is used to denote the typical member of  $B$ , here a consumption bundle that is specified by listing the respective quantities of the two goods. The colon is read as “such that”.<sup>2</sup> Second, after the colon, the three properties that these typical members must satisfy are all listed.

<sup>2</sup> Alternative notation for “such that” is |.

This completes the specification of  $B$ . Indeed, Eq. (1.1.1) is an example of the general specification:

$$S = \{\text{typical member} : \text{defining properties}\}$$

Note that it is not just infinite sets that can be specified by properties like this—finite sets can too. Indeed, some finite sets almost *have* to be specified in this way, such as the set of all human beings currently alive.

## Set Membership

As we stated earlier, sets contain members or elements. Some convenient standard notation is used to express the relation between a set and its members. First,

$$x \in S$$

indicates that  $x$  is an *element* of  $S$ . Note the special “belongs to” symbol  $\in$  (which is a variant of the Greek letter  $\varepsilon$ , or “epsilon”).

To express the fact that  $x$  is *not* a member of  $S$ , we write  $x \notin S$ . For example,  $d \notin \{a, b, c\}$  says that  $d$  is not an element of the set  $\{a, b, c\}$ .

To see how set membership notation can be applied, consider again the example of a first-year college student who *must* buy both a laptop and a smartphone. Suppose that there are two types of each device, “cheap” and “expensive”. Suppose too that the student cannot afford to combine the expensive smartphone with the expensive laptop. Then the set of three combinations that the student can afford is {cheap laptop and cheap smartphone, expensive laptop and cheap smartphone, cheap laptop and expensive smartphone}. Thus, the student is restricted to choosing one of the three combinations in this set. If we denote the choice by  $s$  and the affordable set by  $B$ , we can say that the student’s choice is constrained by the requirement that  $s \in B$ . If we denote by  $t$  the unaffordable combination of an expensive laptop and an expensive smartphone, we can express this unaffordability by writing  $t \notin B$ .

Let  $A$  and  $B$  be any two sets. Set  $A$  is a *subset* of  $B$  if it is true that every member of  $A$  is also a member of  $B$ . When that is the case, we write  $A \subseteq B$ . In particular,  $A \subseteq A$  and  $\emptyset \subseteq A$ . Recall that two sets are *equal* if they contain the same elements. From the definitions, we see that  $A = B$  when, and only when, both  $A \subseteq B$  and  $B \subseteq A$ .

To continue the previous example, suppose that the student can make do with a cheap smartphone, so she chooses not to buy an expensive one. Having made this choice, she only needs to decide which laptop to buy in addition to the cheap smartphone. Let  $A$  denote the set {cheap laptop and cheap smartphone, expensive laptop and cheap smartphone} of options the student has not ruled out. Then we have  $A \subseteq B$ .

## Set Operations

Sets can be combined in many different ways. Especially important are three operations: the *union*, *intersection*, and the *difference* of any two sets  $A$  and  $B$ , as shown in Table 1.1.1.



**Table 1.1.1** Elementary set operations

Notation	Name	The set that consists of:
$A \cup B$	$A$ union $B$	all elements belonging to at least one of the sets $A$ and $B$
$A \cap B$	$A$ intersection $B$	all elements belonging to both $A$ and $B$
$A \setminus B$	$A$ minus $B$	all elements belonging to set $A$ , but not to $B$

In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

It is important to notice that the word “or” in mathematics is *inclusive*, in the sense that the statement “ $x \in A$  or  $x \in B$ ” allows for the possibility that  $x \in A$  and  $x \in B$  are *both* true.

**EXAMPLE 1.1.1** Let  $A = \{1, 2, 3, 4, 5\}$  and  $B = \{3, 6\}$ . Find  $A \cup B$ ,  $A \cap B$ ,  $A \setminus B$ , and  $B \setminus A$ .<sup>3</sup>

**Solution:**  $A \cup B = \{1, 2, 3, 4, 5, 6\}$ ,  $A \cap B = \{3\}$ ,  $A \setminus B = \{1, 2, 4, 5\}$ ,  $B \setminus A = \{6\}$ . ■

As an economic example, considering everybody who worked in California during the year 2019. Let  $A$  denote the set of all those workers who have an income of at least \$35 000 for the year; let  $B$  denote the set of all who have a net worth of at least \$200 000. Then  $A \cup B$  would be those workers who earned at least \$35 000 or who had a net worth of at least \$200 000, whereas  $A \cap B$  are those workers who earned at least \$35 000 and who also had a net worth of at least \$200 000. Finally,  $A \setminus B$  would be those who earned at least \$35 000 but whose net worth was less than \$200 000.

If two sets  $A$  and  $B$  have no elements in common, they are said to be *disjoint*. Thus, the sets  $A$  and  $B$  are disjoint if  $A \cap B = \emptyset$ .

A collection of sets is often referred to as a *family* of sets. When considering a certain family of sets, it is often natural to think of each set in the family as a subset of one particular fixed set  $\mathcal{U}$ , hereafter called the *universal set*. In the previous example, the set of all residents of California in 2019 would be an obvious choice for a universal set.

If  $A$  is a subset of the universal set  $\mathcal{U}$ , then according to the definition of difference,  $\mathcal{U} \setminus A$  is the set of elements of  $\mathcal{U}$  that are not in  $A$ . This set is called the *complement* of  $A$  in  $\mathcal{U}$  and is denoted by  $A^c$ .<sup>4</sup> When finding the complement of a set, it is *very* important to be clear about which universal set is being used.

**EXAMPLE 1.1.2** Let the universal set  $\mathcal{U}$  be the set of all students at a particular university. Among these, let  $F$  denote the set of female students,  $M$  the set of all mathematics students,  $C$  the set of students in the university choir,  $B$  the set of all biology students, and  $T$  the set of all tennis

<sup>3</sup> Here and throughout the book, we often write the examples in the form of exercises. We strongly suggest that you first attempt to solve the problem, while covering the solution, and then gradually reveal the proposed solution to see if you are right.

<sup>4</sup> Other ways of denoting the complement of  $A$  include  $\complement A$  and  $\tilde{A}$ .

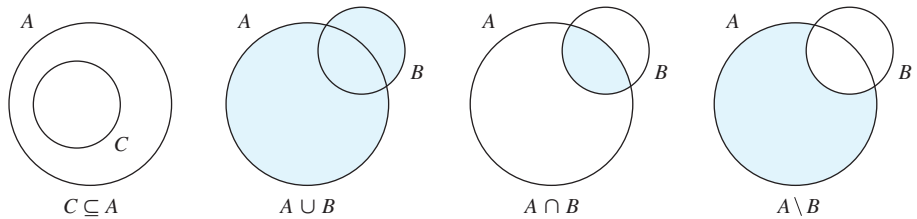
players. Describe the members of the following sets:  $\mathcal{U} \setminus M$ ,  $M \cup C$ ,  $F \cap T$ ,  $M \setminus (B \cap T)$ , and  $(M \setminus B) \cup (M \setminus T)$ .

**Solution:**  $\mathcal{U} \setminus M$  consists of those students who are not studying mathematics,  $M \cup C$  of those students who study mathematics and/or are in the choir. The set  $F \cap T$  consists of those female students who play tennis. The set  $M \setminus (B \cap T)$  has those mathematics students who do not both study biology and play tennis. Finally, the last set  $(M \setminus B) \cup (M \setminus T)$  has those students who either are mathematics students not studying biology or mathematics students who do not play tennis. Can you see that the last two sets must be equal?<sup>5</sup> ■

## Venn Diagrams

When considering how different sets may be related, it is often both instructive and extremely helpful to represent each set by a region in a plane. Diagrams constructed in this manner are called *Venn diagrams*.<sup>6</sup>

For pairs of sets, the definitions discussed in the previous section can be illustrated as in Fig. 1.1.1. By using the definitions directly, or by illustrating sets with Venn diagrams, one can derive formulas that are universally valid regardless of which sets are being considered. For example, the formula  $A \cap B = B \cap A$  follows immediately from the definition of the intersection between two sets.



**Figure 1.1.1** Four Venn diagrams

When dealing with three general sets  $A$ ,  $B$ , and  $C$ , it is important to draw the Venn diagram so that all possible relations between an element and each of the three sets are represented. In other words, as in Fig. 1.1.3, the following eight different regions should all be nonempty:<sup>7</sup>

- |                             |                             |                             |                             |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1. $(A \cap B) \setminus C$ | 2. $(B \cap C) \setminus A$ | 3. $(C \cap A) \setminus B$ | 4. $A \setminus (B \cup C)$ |
| 5. $B \setminus (C \cup A)$ | 6. $C \setminus (A \cup B)$ | 7. $(A \cap B) \cap C$      | 8. $((A \cup B) \cup C)^c$  |

<sup>5</sup> For arbitrary sets  $M$ ,  $B$ , and  $T$ , it is true that  $(M \setminus B) \cup (M \setminus T) = M \setminus (B \cap T)$ . It should become easier to verify this equality after you have studied the following discussion of Venn diagrams.

<sup>6</sup> Named after the English mathematician John Venn (1834–1923), who was the first to use them extensively.

<sup>7</sup> That is, all should be nonempty unless something more is known about the relation between the three sets. For example, one might have specified that the sets must be disjoint, meaning that  $A \cap B \cap C = \emptyset$ . In this case region (7) in Fig. 1.1.3 disappears.

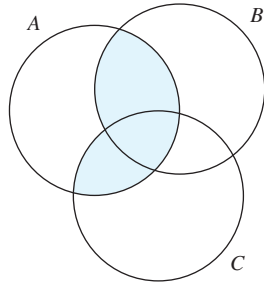
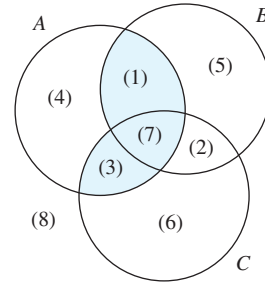
Figure 1.1.2 Venn diagram for  $A \cap (B \cup C)$ 

Figure 1.1.3 Venn diagram for three sets

Venn diagrams are particularly useful when limited to no more than three sets. For instance, consider the following possible relationship between the three sets  $A$ ,  $B$ , and  $C$ :

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1.1.2)$$

Using only the definitions in Table 1.1.1, it is somewhat difficult to verify that Eq. (1.1.2) holds for all sets  $A$ ,  $B$ ,  $C$ . Using a Venn diagram, however, it is easily seen that the two sets on the left- and right-hand sides of (1.1.2) are both represented by the region made up of the three regions that are shaded in both Fig. 1.1.2 and Fig. 1.1.3. This confirms Eq. (1.1.2). Similar reasoning allows one to prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (1.1.3)$$

Using either the definition of intersection and union or appropriate Venn diagrams, one can see that  $A \cup (B \cup C) = (A \cup B) \cup C$  and that  $A \cap (B \cap C) = (A \cap B) \cap C$ . Consequently, in such cases it does not matter where the parentheses are placed, so they can be dropped and the expressions written as  $A \cup B \cup C$  and  $A \cap B \cap C$ . That said, note that the parentheses cannot generally be removed in the two expressions on the left-hand sides of Eqs (1.1.2) and (1.1.3). This is because  $A \cap (B \cup C)$  is generally not equal to  $(A \cap B) \cup C$ , and  $A \cup (B \cap C)$  is generally not equal to  $(A \cup B) \cap C$ .<sup>8</sup>

Notice, however, that this way of representing sets in the plane becomes unmanageable if four or more sets are involved. This is because a Venn diagram with, for example, four sets would have to contain  $2^4 = 16$  regions.<sup>9</sup>

## Georg Cantor

The founder of set theory is Georg Cantor (1845–1918), who was born in Saint Petersburg but moved to Germany at the age of eleven. He is regarded as one of history's great mathematicians. This is not because of his contributions to the development of the useful, but relatively trivial, aspects of set theory outlined above. Rather, Cantor is remembered for his profound study of infinite sets. Below we try to give just a hint of his theory's implications.

<sup>8</sup> For practice, demonstrate this fact by considering the case where  $A = \{1, 2, 3\}$ ,  $B = \{2, 3\}$ , and  $C = \{4, 5\}$ , or by using a Venn diagram.

<sup>9</sup> One can show that a Venn diagram with  $n$  sets would have to contain  $2^n$  regions.

A collection of individuals are gathering in a room that has a certain number of chairs. How can we find out if there are exactly as many individuals as chairs? One method would be to count the chairs and count the individuals, and then see if they total the same number. Alternatively, we could ask all the individuals to sit down. If they all have a seat to themselves and there are no chairs unoccupied, then there are exactly as many individuals as chairs. In that case each chair corresponds to an individual and each individual corresponds to a chair—i.e., there is a “one-to-one correspondence” between individuals and chairs.

Generally mathematicians say that two sets of elements have the same *cardinality*, if there is a one-to-one correspondence between the sets. This definition is also valid for sets with an infinite number of elements. Cantor struggled for three years to prove a surprising implication of this definition—that there are as many points in a square as there are points on one of its edges of the square, in the sense that the two sets have the same cardinality.<sup>10</sup>

### EXERCISES FOR SECTION 1.1

1. Let  $A = \{2, 3, 4\}$ ,  $B = \{2, 5, 6\}$ ,  $C = \{5, 6, 2\}$ , and  $D = \{6\}$ .
  - (a) Determine which of the following six statements are true:  $4 \in C$ ;  $5 \in C$ ;  $A \subseteq B$ ;  $D \subseteq C$ ;  $B = C$ ; and  $A = B$ .
  - (b) List all members of each of the following eight sets:  $A \cap B$ ;  $A \cup B$ ;  $A \setminus B$ ;  $B \setminus A$ ;  $(A \cup B) \setminus (A \cap B)$ ;  $A \cup B \cup C \cup D$ ;  $A \cap B \cap C$ ; and  $A \cap B \cap C \cap D$ .
2. Let  $F$ ,  $M$ ,  $C$ ,  $B$ , and  $T$  be the sets in Example 1.1.2.
  - (a) Describe the following sets:  $F \cap B \cap C$ ,  $M \cap F$ , and  $((M \cap B) \setminus C) \setminus T$ .
  - (b) Write the following statements in set terminology:
    - (i) All biology students are mathematics students.
    - (ii) There are female biology students in the university choir.
    - (iii) No tennis player studies biology.
    - (iv) Those female students who neither play tennis nor belong to the university choir all study biology.
3. A survey revealed that 50 people liked coffee and 40 liked tea. Both these figures include 35 who liked both coffee and tea. Finally, ten did not like either coffee or tea. How many people in all responded to the survey?
4. Make a complete list of all the different subsets of the set  $\{a, b, c\}$ . How many are there if the empty set and the set itself are included? Do the same for the set  $\{a, b, c, d\}$ .
5. Determine which of the following formulas are true. If any formula is false, find a counter example to demonstrate this, using a Venn diagram if you find it helpful.

<sup>10</sup> In 1877, in a letter to German mathematician Richard Dedekind (1831–1916), Cantor wrote of this result: “I see it, but I do not believe it.”